Proportional resource allocation

Alexandros Voudouris

University of Oxford

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  - Processing time of a CPU
  - Storage space of a cloud

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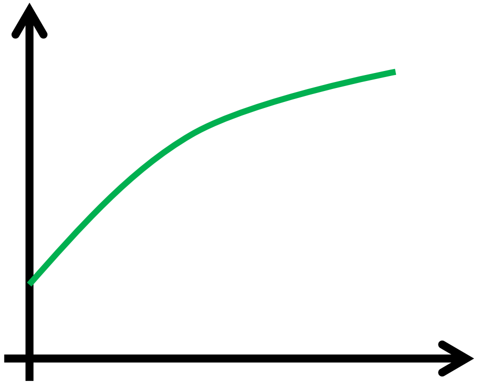
- n players with **valuation functions** $v_i: [0,1] \rightarrow \mathbb{R}_{\geq 0}$
  - $v_i(x)$ represents the value of user i for a fraction x of the resource
  - concave
  - increasing
  - differentiable



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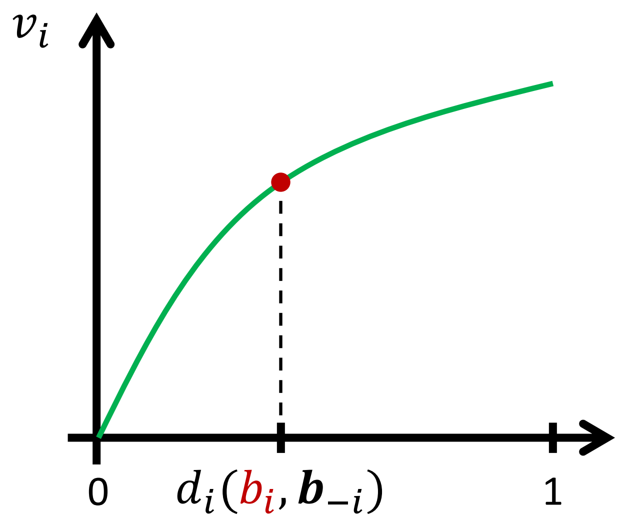
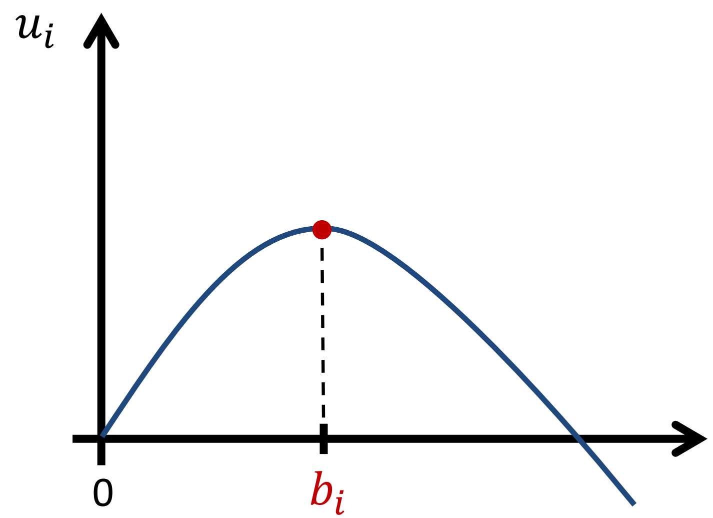
- The **utility** of each player is defined as the difference between the value she has for the fraction she receives, minus her payment:

$$u_i(\boldsymbol{b}) = v_i(d_i(\boldsymbol{b})) - b_i$$

The game

- Since $v_i(x)$ is a concave function, $u_i(\boldsymbol{b})$ is a concave function





Best response computation

- $B_{-i} = \sum_{j \neq \mathrm{i}} b_j$
- Compute the utility derivative of player i as function of her generic bid y:

$$\frac{\partial u_i(y, \boldsymbol{b}_{-i})}{\partial y} = \left(v_i \left(\frac{y}{y + B_{-i}} \right) - y \right)'$$
$$= \frac{B_{-i}}{(y + B_{-i})^2} v_i' \left(\frac{y}{y + B_{-i}} \right) - 1$$

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- If the derivative is negative for every y, then $b_i = 0$
- Otherwise, b_i is the solution of the equation that is derived by nullifying the derivative

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- Nullify the utility derivatives:

$$\left(\frac{b_1}{b_1 + b_2} - b_1\right)' = 0 \Leftrightarrow b_2 = (b_1 + b_2)^2$$

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$$b_1 = \frac{2}{9}$$

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- Actually, we aim to maximize the sum of utilities, but considering the payments as the total utility of the resource owner, the social welfare definition gets simplified

Efficiency at equilibrium

- Refine price of anarchy for utility maximization:

$$\text{PoA} = \max_{\boldsymbol{b} \in \text{NE}} \frac{\text{SW}(\boldsymbol{d}_{\boldsymbol{OPT}})}{\text{SW}(\boldsymbol{d}(\boldsymbol{b}))}$$

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- $\boldsymbol{d_{OPT}}$ is the allocation achieving the maximum possible social welfare
- Note the difference from cost minimization games: we take the optimal social welfare over the social welfare of the equilibrium
- Since the equilibrium is unique, the max operator doesn't make any difference for the price of anarchy of a given game

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- To compute the optimal allocation, we just need to look at the linear functions and give the whole resource to player 1 who has the **largest slope** ⇨ $\mathrm{SW}(d_{OPT}) = v_1(1) = 1$
- Hence, the price of anarchy of this game is 6/5

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Theorem
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- $\boldsymbol{b}$ = bids of all players at equilibrium
- $B_{-i} = \sum_{j \neq \mathrm{i}} b_j$
- $B = b_i + B_{-i}$
- d_i = resource fraction player i gets at equilibrium
- x_i = optimal resource fraction of player i

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&\geq \frac{1}{2} v_i(x_i) - x_i B
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$\Rightarrow$ POA ≤ 2

$\square$

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- At equilibrium:
  - The first player gets half the resource for a value of $1/2$
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- The optimal allocation is to give the whole resource to the first player for a value of 1 $\square$

Can we fill in the gap?

- We know an upper bound of 2 and a lower bound of $4/3$
- Two possible ways to go:
  - either try to improve the lower bound by finding a different example with worst price of anarchy,
  - or try to decrease the upper bound by taking the equilibrium into account

Worst-case games

Lemma

For any $\boldsymbol{b}$,

$$\frac{\text{SW}(\boldsymbol{d}_{\boldsymbol{OPT}})}{\text{SW}(\boldsymbol{d}(\boldsymbol{b}))} \le \frac{\max_i\left\{v_i'\big(d_i(\boldsymbol{b})\big)\right\}}{\sum_i d_i(\boldsymbol{b}) \cdot v_i'\big(d_i(\boldsymbol{b})\big)}$$

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- Similarly:

$$\sum_i d_i v_i'(d_i) \leq \max_i\{v_i'(d_i)\}$$

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$$\frac{\mathrm{SW}(\boldsymbol{d}(\boldsymbol{b}))}{\mathrm{SW}(\boldsymbol{d_{OPT}})} \leq \frac{\sum_i \left(v_i(d_i) - d_i v_i'(d_i)\right) + \max_i\{v_i'(d_i)\}}{\sum_i \left(v_i(d_i) - d_i v_i'(d_i)\right) + \sum_i d_i v_i'(d_i)}$$

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- The lemma follows by applying the inequality:

$$\frac{\alpha + \beta}{\alpha + \gamma} \leq \frac{\beta}{\gamma}, \qquad \forall \alpha \geq 0, b \geq \gamma$$

with $\alpha = \sum_i \big(v_i(d_i) - d_i v_i'(d_i)\big)$, $\beta = \max_i \{v_i'(d_i)\}, \gamma = \sum_i d_i v_i'(d_i)$

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How can we exploit this?

$$\frac{\mathrm{SW}(\boldsymbol{d_{OPT}})}{\mathrm{SW}(\boldsymbol{d}(\boldsymbol{b}))} \leq \frac{\max_i \{v_i'(d_i(\boldsymbol{b}))\}}{\sum_i d_i(\boldsymbol{b}) \cdot v_i'(d_i(\boldsymbol{b}))}$$

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- This inequality indicates that for every resource allocation game with increasing concave valuation functions, there exists another game with worse price of anarchy such that
  - every player has a linear valuation function with slope equal to the valuation derivate at equilibrium in the original game, and
  - the optimal allocation is such that the whole resource is shared between the players with maximum slope

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Theorem
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- $B_{-i} = \sum_{j \neq \mathrm{i}} b_j$
- $B = b_i + B_{-i}$
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- Only consider the players with positive bids, everyone else gets zero fraction of the resource and does not contribute to the social welfare

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- If $\frac{3}{4}\alpha_i - B < 0$, the inequality holds trivially

A tight PoA bound

- Definition of social welfare at equilibrium:

$$\mathrm{SW}(\boldsymbol{d}) = \sum_{i \in N} v_i(d_i) = \sum_{i \in N} (u_i(\boldsymbol{b}) + b_i)$$

$$u_i(\boldsymbol{b}) \geq \frac{3}{4} v_i(x_i) - x_i B$$

$$\geq \sum_{i \in N} \left(\frac{3}{4} v_i(x_i) - x_i B \right) + B$$

$$= \frac{3}{4} \sum_{i \in N} v_i(x_i) - B \sum_{i \in N} x_i + B$$

$$\sum_{i \in N} x_i = 1$$

$$\geq \frac{3}{4} \mathrm{SW}(\boldsymbol{d_{OPT}})$$

$$\Rightarrow \text{POA} \leq \frac{4}{3}$$

□

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- **Worst case for PoA:** all players have linear valuation functions
- The price of anarchy is at most $4/3$ and this bound is tight

Some further readings

- **Efficiency loss in a network resource allocation game**
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  - Mathematics of Operations Research, vol. 29(3):407–435, 2004
- **Optimal allocation of a divisible good to strategic buyers**
  - S. Sanghavi and B. Hajek
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- **Welfare guarantees for proportional allocations**
  - I. Caragiannis and A. A. Voudouris
  - Theory of Computing Systems, vol. 59(4):581–599, 2016