Mechanism design: Single-parameter environments

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Single-item auctions

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- n **agents**

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  - This value represents the **willingness-to-pay** of the agent; that is, v_i is the maximum amount of money that agent i is willing to pay in order to buy the item
- The utility of each agent is **quasilinear in money**:
  - If agent i loses the item, then her utility is 0
  - If agent i wins the item at price p, then her utility is $v_i - p$

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  - **Truthful auctions** that **maximize the social welfare**

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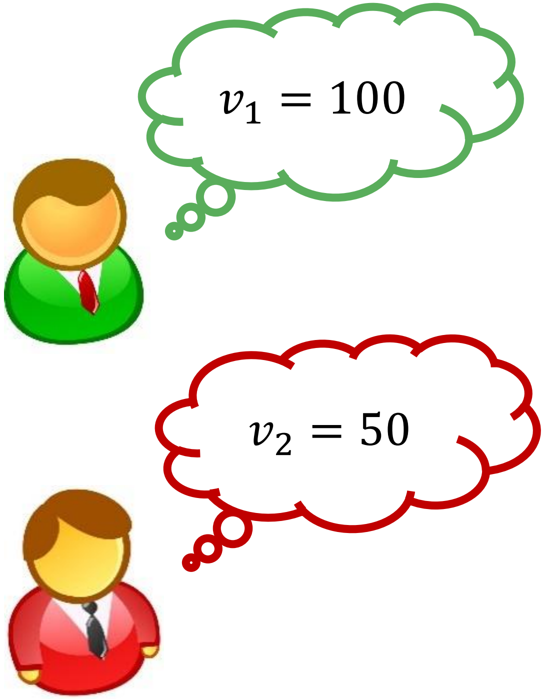

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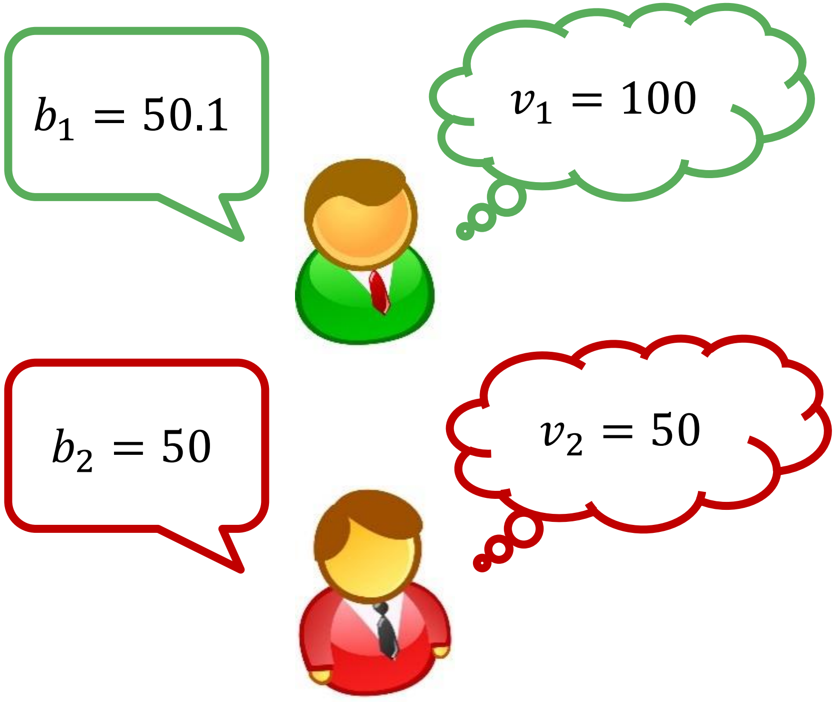

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In a second-price auction
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- (b) is obvious:
  - the selling price is at most the winner's bid, and the bid of a truthtelling bidder is equal to her true value

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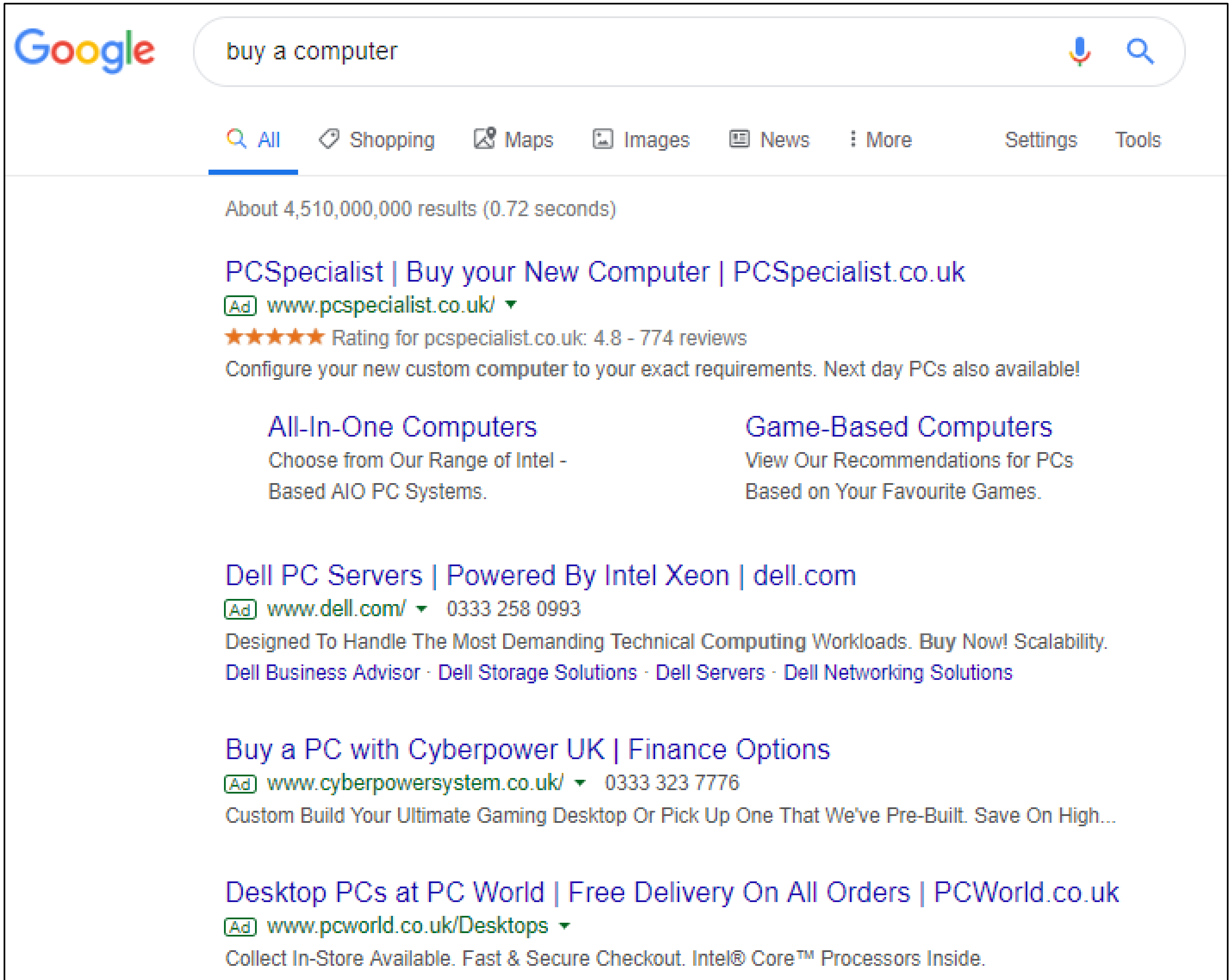
Case II: $v_i \geq B$

- Maximum possible utility $= v_i - B$
- Bidder i wins the item by setting $b_i = v_i$ $\square$

Sponsored search auctions



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- Each bidder i has a **private value** v_i **per click**
  - Bidder i derives utility $a_j \cdot v_i$ from slot j

Sponsored search auctions: goals

- **Truthfulness:** It is a dominant strategy for each bidder to bid her true value
- **Social welfare maximization:** $\sum_i v_i \cdot x_i$
  - x_i is the CTR of the slot that bidder i is assigned to, or 0 otherwise
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- Can we extend the ideas we exploited for single-item auctions?

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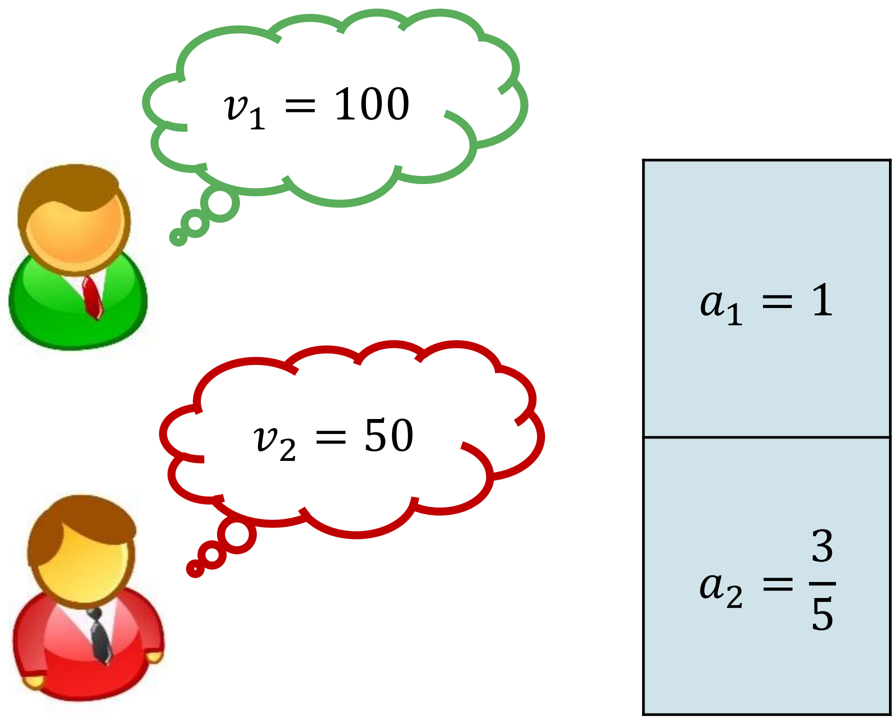

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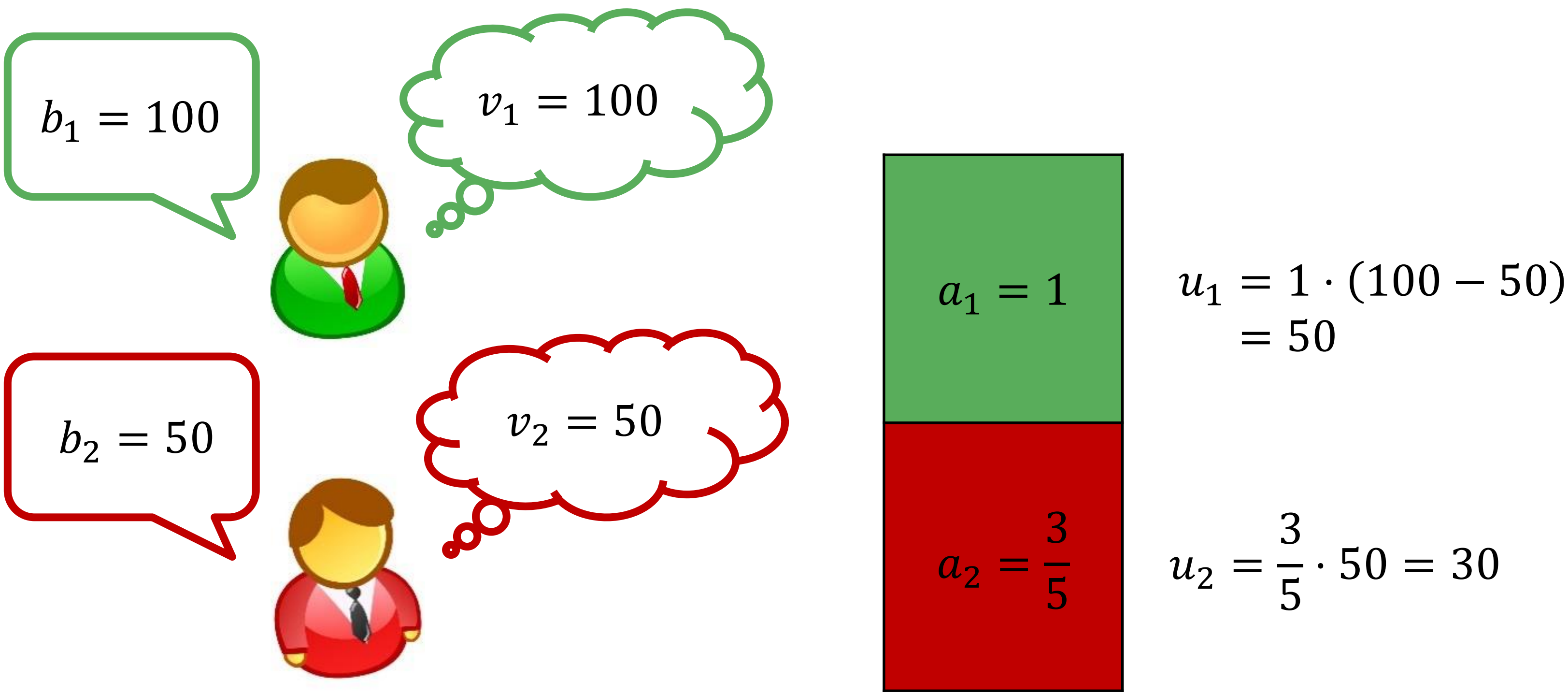

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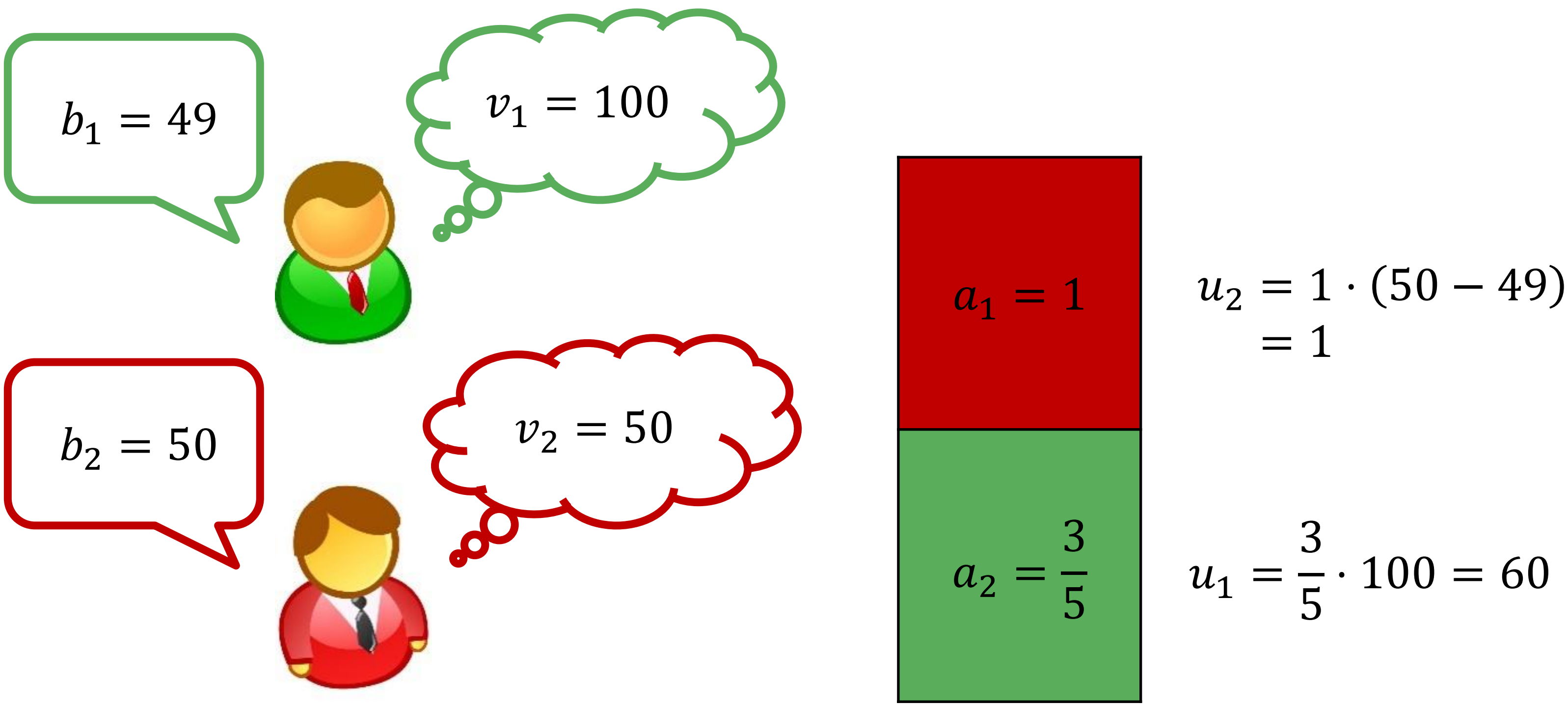

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- Focus on payment rules such that $p_i(\boldsymbol{b}) \in [0, b_i \cdot x_i(\boldsymbol{b})]$
  - $p_i(\boldsymbol{b}) \geq 0$ ensures that the seller does not pay the bidders
  - $p_i(\boldsymbol{b}) \leq b_i \cdot x_i(\boldsymbol{b})$ ensures non-negative utility for truthful bidders

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Lemma [Myerson, 1981]

(a) An allocation rule $\boldsymbol{x}$ is implementable if and only if it is monotone

(b) For every allocation rule $\boldsymbol{x}$, there exists a unique payment rule $\boldsymbol{p}$ such that $(\boldsymbol{x}, \boldsymbol{p})$ is a truthful auction

Proof of Myerson's Lemma

- Fix a bidder i, and the bids $\boldsymbol{b}_{-i}$ of the other bidders
- Given that these quantities are now fixed, we simplify our notation:
  - $x(z) = x_i(z, \boldsymbol{b}_{-i})$
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- The idea:
  - assuming $(\boldsymbol{x}, \boldsymbol{p})$ is a truthful auction, the bidder has no incentive to unilaterally deviate to any other bid
  - This will give us a relation between $\boldsymbol{x}$ and $\boldsymbol{p}$, which we can use to derive an explicit formula for $\boldsymbol{p}$ as a function of $\boldsymbol{x}$

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$$u(y) \ge u(z) \Leftrightarrow y \cdot x(y) - p(y) \ge y \cdot x(z) - p(z)$$
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- Combining these two, we get:

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  ⇨ (a) is now proved

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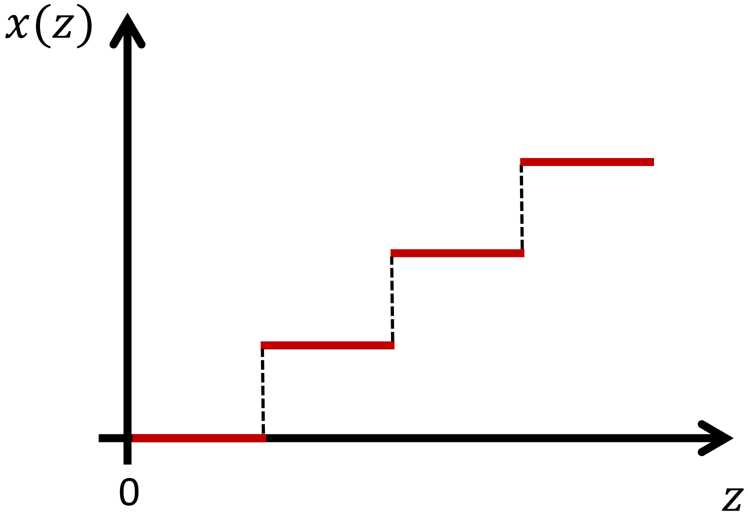
- Assume $\boldsymbol{x}$ is piecewise constant, like in sponsored search auctions



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$$\text{jump of } p \text{ at } z = z \cdot (\text{jump of } x \text{ at } z)$$

- Therefore, we can define the payment of the bidder as

$$p(b) = \sum_{y \in [0,b]} y \cdot (\text{jump of } x \text{ at } y)$$

where y enumerates all break points of x in $[0, b]$

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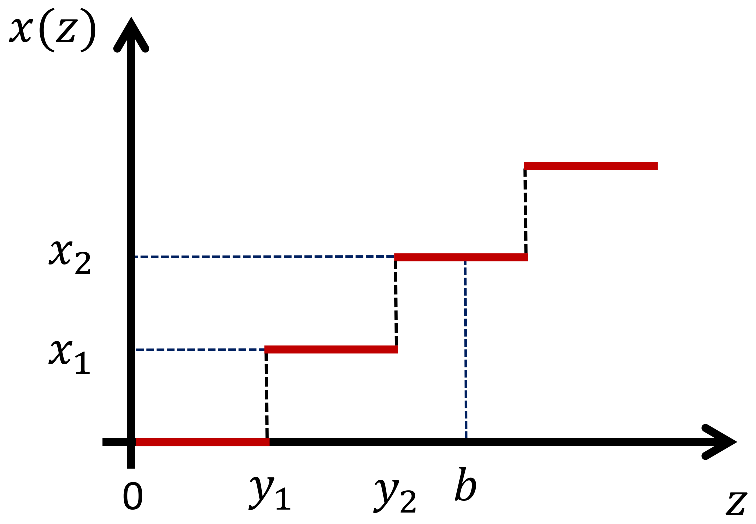
- Example:



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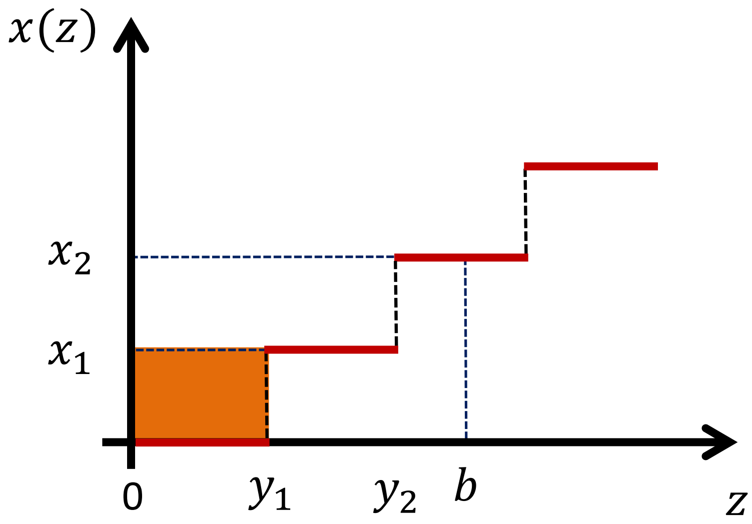

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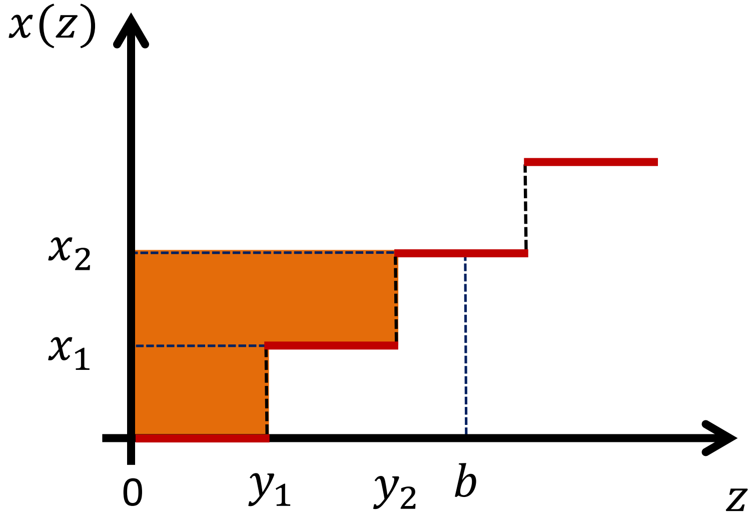

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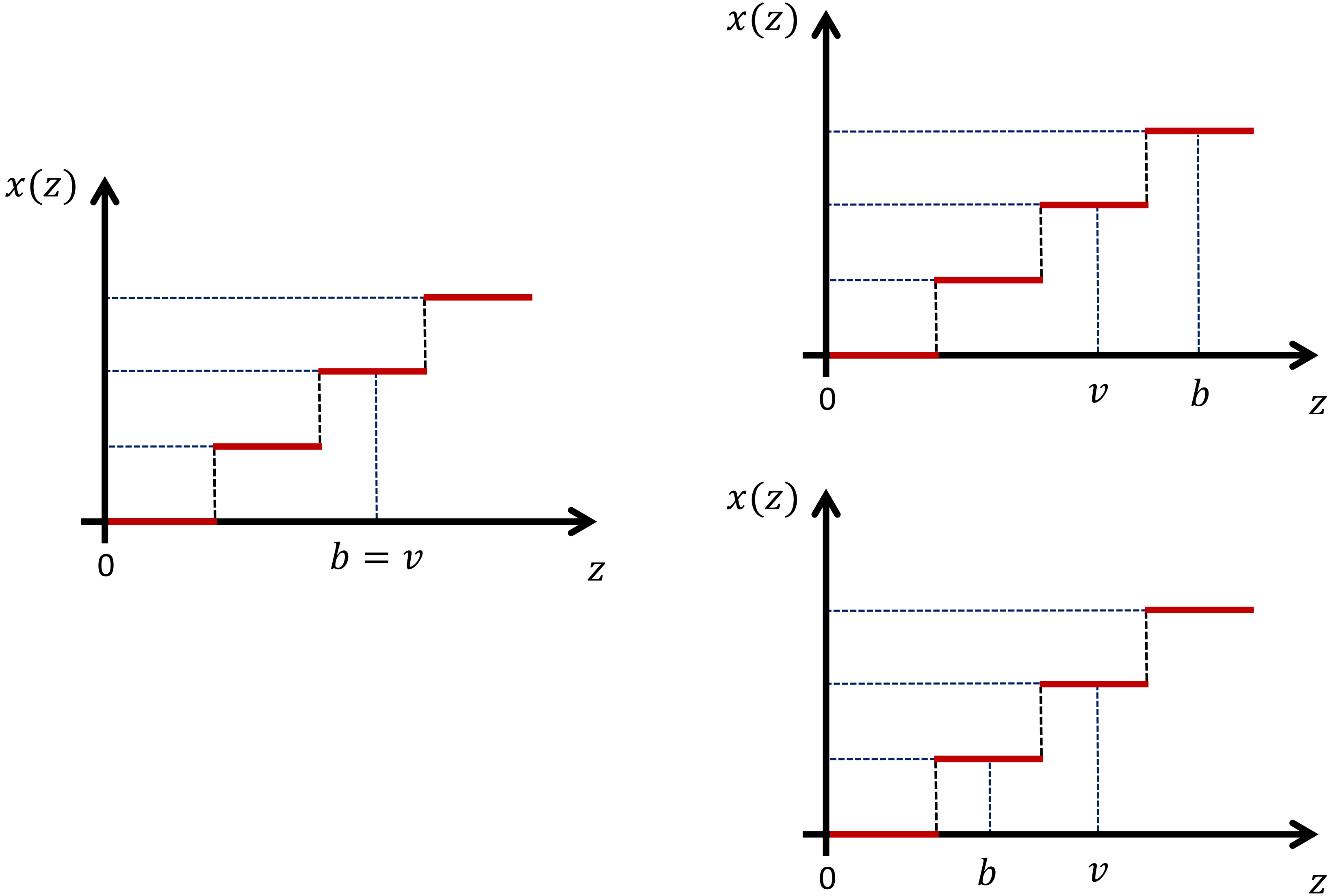

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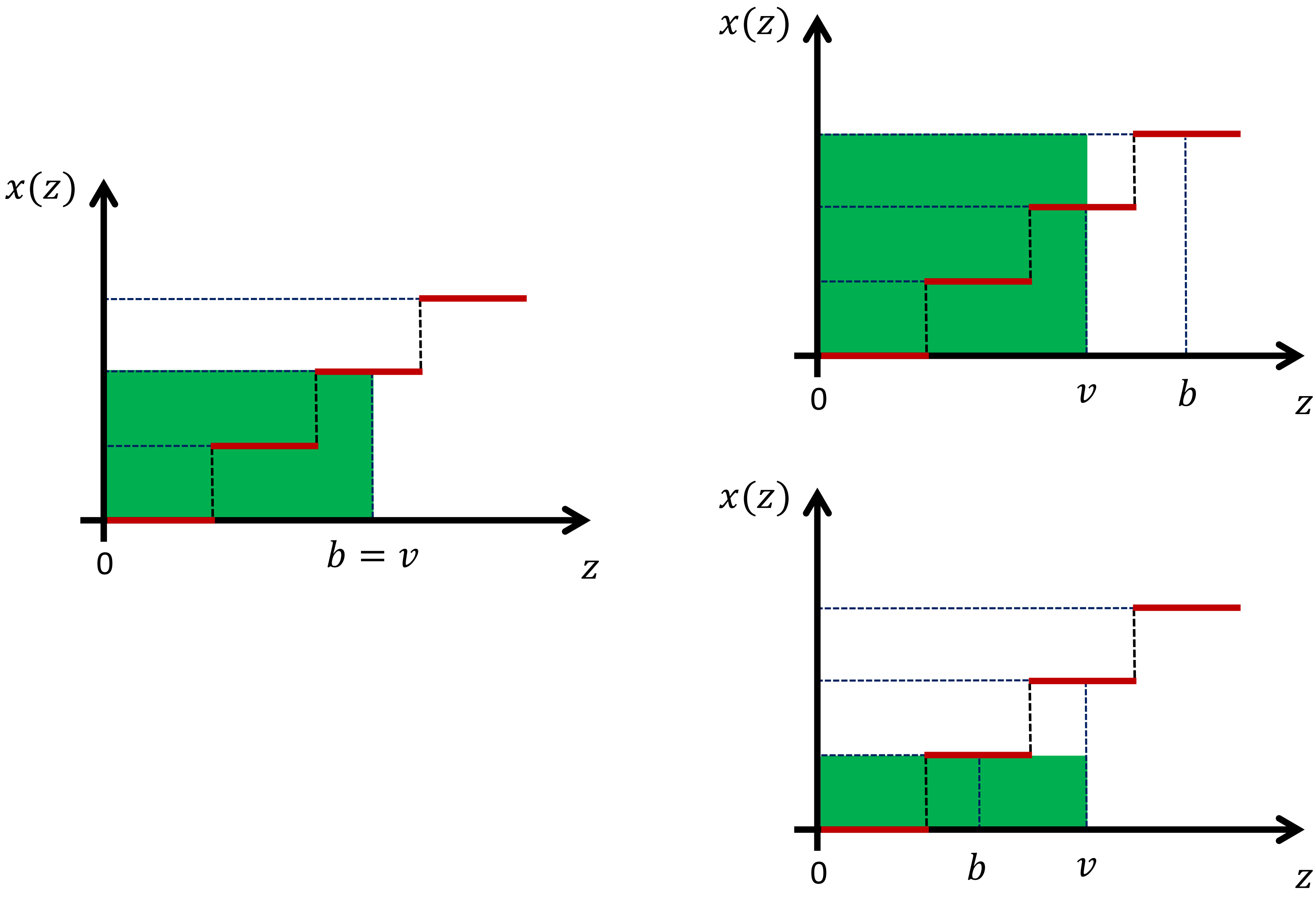

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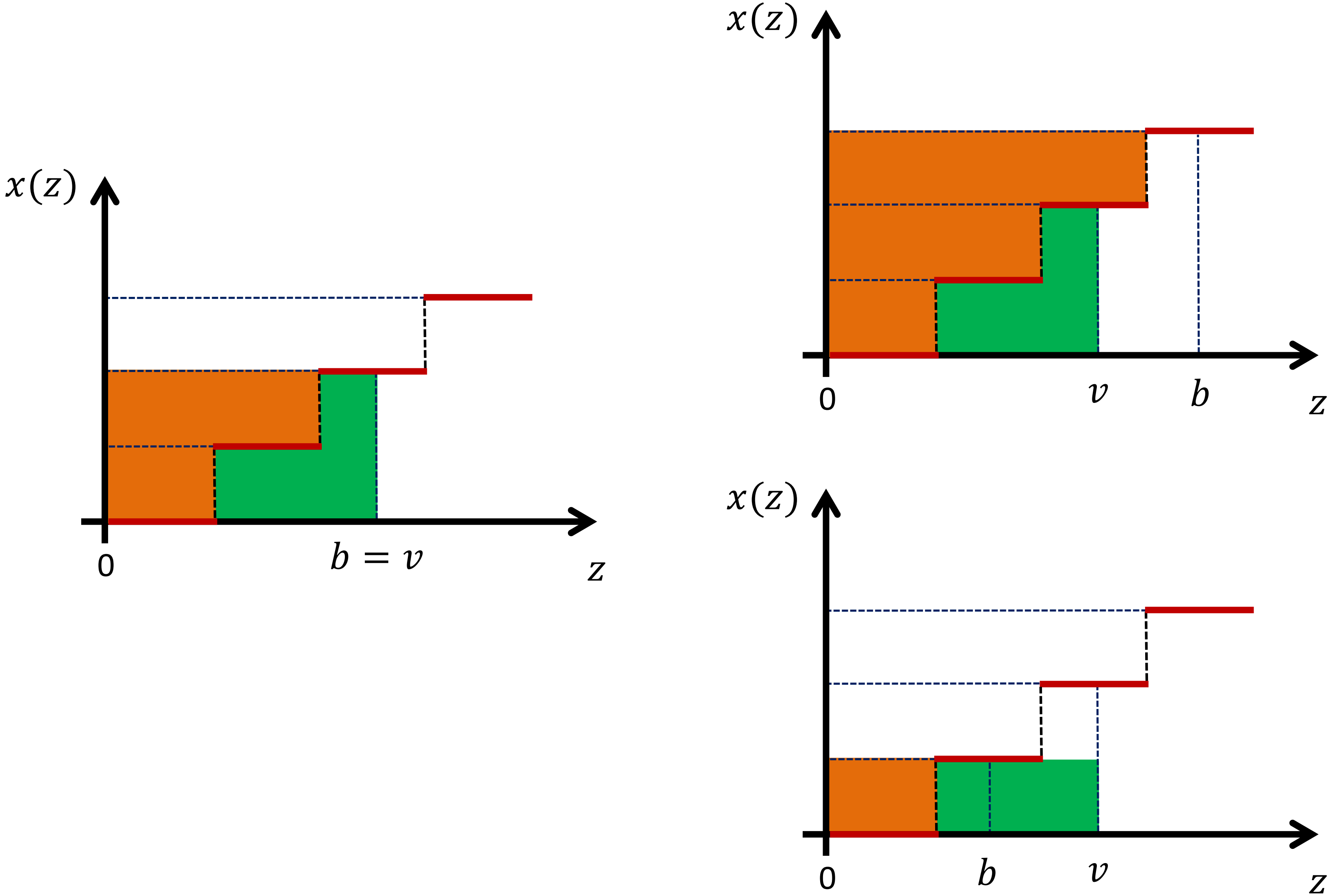

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- jump of x at y: the difference in CTR between two consecutive slots
- The total payment of the i-th highest bidder is:

$$p_i(b_i, \boldsymbol{b}_{-i}) = \sum_{j=i}^{k} b_{j+1}(a_j - a_{j+1})$$

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- **Myerson's Lemma:** a characterization of truthful mechanisms in single-parameter environments

Summary

- **Auctions:** allocation rule + payment rule
- An allocation rule is **implementable** is there exists a payment rule, so that together they define a truthful auction
- An allocation rule is **monotone**, if larger bids give more stuff
- **Single-item auctions:** first-price is not truthful, second-price is truthful and maximizes the social welfare (sells to the bidder with the highest value)
- **Sponsored search auctions:** generalized second-price auction is not truthful
- **Myerson's Lemma:** a characterization of truthful mechanisms in single-parameter environments
- Using Myerson's Lemma we can design a truthful sponsored search auction