Mechanism design: Multi-parameter environments

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- **Our goals:**
  - Incentivize the agents **to truthfully report their values**
  - Choose an outcome that **maximizes the social welfare**

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- In general, the agents might have different values for the possible winners of the item

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  - Each agent i has 2^m parameters

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- **Payment rule:** For a set of functions $h_1, \dots, h_n$ such that h_i is independent of the bid of agent i,

$$p_i(\boldsymbol{b}) = h_i(\boldsymbol{b}_{-i}) - \sum_{j \neq i} b_j\big(\boldsymbol{x}(\boldsymbol{b})\big)$$

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&= \underbrace{v_i\big(\boldsymbol{x}(\boldsymbol{b})\big) + \sum_{j \neq i} b_j\big(\boldsymbol{x}(\boldsymbol{b})\big)}_{\text{The social welfare according to the true value of agent } i \text{ and the bids of the other agents}} - h_i(\boldsymbol{b}_{-i})
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- Agent i cares about the welfare of all agents (based on the reported valuations) and aims to maximize the quantity

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- The mechanism is designed so that the incentives of the agents are aligned with the goal of maximizing the social welfare $\square$

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- There are a lot of different VCG mechanisms, depending on how we choose the h-functions
- We would like to have reasonable payment rules, that satisfy a couple of properties:
  - **Individual rationality:** Every agent has non-negative utility, and therefore incentive to participate
  - **No positive transfers:** The mechanism does not pay the agents, the agents pay the mechanism

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- The payment of agent i is the difference between the maximum social welfare of the other agents when she does not participate, and the social welfare when she participates
- Agent i pays the loss in welfare due to her participation

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Drawbacks of VCG mechanisms

- **Preference elicitation:** VCG mechanisms demand from each agent to communicate her values for every possible outcome
  - Not practical in many situations: communicating 2^m parameters in the case of combinatorial auctions is impossible, even for small m

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- **Social welfare maximization might be a hard problem**
- Knapsack auctions:
  - each agent i demands w_i items and has a private value v_i
  - the seller has a total amount of W items
  - Even though every agent has only one private parameter, maximizing the social welfare is equivalent to the Knapsack problem, which is NP-hard