A utilitarian view of voting rules

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- Each agent $i \in N$ has a **value** v_{ix} for every alternative $x \in A$
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- **Valuation profile:** $\boldsymbol{v} = (v_{ix})_{i \in N,\, x \in A}$

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- The values of an agent i for the alternatives define a ranking $\succ_i$ over them such that $x \succ_i y$ when $v_{ix} \geq v_{iy}$
  – Ties are broken according to some (fixed) tie-breaking rule
- **Ordinal profile** induced by a valuation profile: $\succ_{\boldsymbol{v}} = (\succ_i)_{i \in N}$

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agent	a	b	c	d
1	0.75	0.15	0.07	0.03
2	0.25	0.15	0.2	0.4
3	0.1	0	0.4	0.5
4	0.21	0.3	0.2	0.29

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agent	ranking			
1	a	b	c	d
2	d	a	c	b
3	d	c	a	b
4	b	d	a	c

Social welfare and voting rules

- Given a valuation profile $\boldsymbol{v}$, the **social welfare** of an alternative $x \in A$ is defined as the total value of all agents for x:

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- The social welfare is a measure of how good an alternative is for the society ⇨ **our goal is to choose the alternative with maximum social welfare**
- If we had access to the valuation profile, we could obviously make the optimal social choice
- But … choices are made by voting rules that have access only to the ordinal profile, and therefore electing the optimal alternative is not an easy task

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- The **distortion** of R is the worst-case ratio (over all valuation profiles) between the maximum social welfare (achieved by any alternative) and the social welfare of the alternative chosen by R

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- In general however: $\mathrm{dist}(R) \geq 1$
- We are interested in bounding the distortion of voting rules, and we want these bounds to be as small as possible

A first lower bound

Theorem
The distortion of any deterministic voting rule is $\Omega(m)$

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# agents	ranking				
$m/2$	x	y	a_1	...	a_{m-2}
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- R will choose either alternative x or alternative y
- All other alternatives are dominated by these two alternatives

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$$\mathrm{SW}(x|\boldsymbol{v}) = \frac{m}{2} \cdot \frac{1}{m} = \frac{1}{2}$$

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$\square$

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- Instance with $n = m(m-2)$ agents
- Alternatives $A = \{x, y, a_1, \ldots, a_{m-2}\}$

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- Instance with $n = m(m-2)$ agents
- Alternatives $A = \{x, y, a_1, \dots, a_{m-2}\}$

- For every $j \in [m-2]$, alternative a_i appears first in m rankings
- Alternative x appears second in $\frac{n}{2} = \Theta(m^2)$ rankings
- Alternative y appears second in $\frac{n}{2} = \Theta(m^2)$ rankings
- All agents that rank first the same alternative a_j, rank second either x or y

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- Valuation profile $\boldsymbol{v}$:
  - The m agents that rank a_j first have value $1/m$ for all alternatives; assume these agents rank x second
  - All other agents have value $1/2$ for the alternatives they rank at the first two positions

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- **Case I:** The voting rule chooses alternative a_j for some $j \in [m-2]$
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$$\left.\begin{array}{l} \mathrm{SW}(a_j|\boldsymbol{v}) = m \cdot \dfrac{1}{m} = 1 \\[2em] \mathrm{SW}(y|\boldsymbol{v}) = \Theta(m^2) \cdot \dfrac{1}{2} = \Theta(m^2) \end{array}\right\} \quad \mathrm{dist}(R) = \Omega(m^2)$$

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$$\left.\begin{aligned} \mathrm{SW}(x|\boldsymbol{v}') &= 0 \\ \mathrm{SW}(y|\boldsymbol{v}') &= 0 \\ \mathrm{SW}(z|\boldsymbol{v}') &> 0, \forall z \neq x, y \end{aligned}\right\} \quad \mathrm{dist}(R) \text{ is unbounded}$$

$\square$

An asymptotically tight upper bound

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There exists a voting rule with distortion $O(m^2)$

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- The winner x must be ranked first at least n/m times
- The corresponding agents must have value at least $1/m$ for x

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- **Plurality rule**
- The winner x must be ranked first at least n/m times
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$$\left.\begin{array}{l} \mathrm{SW}(x|\boldsymbol{v}) \geq \dfrac{n}{m} \cdot \dfrac{1}{m} = \dfrac{n}{m^2} \\[2ex] \mathrm{SW}(y|\boldsymbol{v}) \leq n \end{array}\right\} \quad \mathrm{dist}(\mathbf{PL}) = O(m^2)$$

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Randomized voting rules

- A randomized voting rule R defines a probability distribution p_R over the alternatives according to which the winning alternative is chosen

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- The efficiency of R is now measured by the expected social welfare of the winner:

$$\mathbb{E}[\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})] = \sum_{x \in A} p_R(x) \cdot \mathrm{SW}(x|\boldsymbol{v})$$

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- Refinement of distortion:

$$\mathrm{dist}(R) = \sup_{\boldsymbol{v}} \frac{\max_{x \in A} \mathrm{SW}(x|\boldsymbol{v})}{\mathbb{E}[\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})]}$$

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- Harmonic scoring rule: $\mathbf{H} = (1, 1/2, \dots, 1/m)$
- $sc(x)$ = score of alternative x according to $\mathbf{H}$

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<u>Theorem</u>

There exists a randomized voting rule with distortion $O\left(\sqrt{m \cdot \ln m}\right)$

- Harmonic scoring rule: $\mathbf{H} = (1, 1/2, \dots, 1/m)$
- $sc(x)$ = score of alternative x according to $\mathbf{H}$
- **Voting rule:**
  - Rule 1: Choose alternative x with probability $\frac{\mathrm{sc}\,(x)}{\sum_{y \in A} \mathrm{sc}(y)}$
  - Rule 2: Choose alternative x with probability $1/m$
  - Run the two rules with probability $1/2$ each

An improved distortion bound

- Let x be the optimal alternative
- We distinguish between two cases, depending on the harmonic score of x
  - **Case I:** $\mathrm{sc}(x) \geq n \cdot \sqrt{\frac{\ln m+1}{m}}$
  - **Case II:** $\mathrm{sc}(x) < n \cdot \sqrt{\frac{\ln m+1}{m}}$

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- $p_R(x) \geq \frac{1}{2} \cdot \frac{\mathrm{SC}(x)}{\sum_{y \in \mathrm{A}} \mathrm{SC}(y)} \geq \frac{1}{2} \cdot \frac{n \cdot \sqrt{\frac{\ln m+1}{m}}}{n\,(\ln m+1)} = \frac{1}{2\sqrt{m(\ln m+1)}}$

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$$\mathbb{E}[\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})] \geq p_R(x) \cdot \mathrm{SW}(x|\boldsymbol{v}) \geq \frac{\mathrm{SW}(x|\boldsymbol{v})}{2\sqrt{m(\ln m+1)}}$$

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$$\Rightarrow \mathrm{dist}(R) \leq 2\sqrt{m(\ln m + 1)}$$

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Case II: $sc(x) < n \cdot \sqrt{\frac{\ln m+1}{m}}$

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- If alternative x is ranked k–th by agent i, then $v_{ix} \leq \frac{1}{k}$
  ⇨ $\mathrm{SW}(x|\boldsymbol{v}) \leq \mathrm{sc}(x)$

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$$\Rightarrow \mathrm{dist}(R) = \frac{\mathrm{SW}(x|\boldsymbol{v})}{\mathbb{E}[\mathrm{SW}(R(\succ_{\boldsymbol{v}})|\boldsymbol{v})]} \leq \frac{n \cdot \sqrt{\frac{\ln m + 1}{m}}}{\frac{n}{2m}} = 2\sqrt{m(\ln m + 1)}$$

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Best known bounds

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- We can actually do a little bit better than $O\left(\sqrt{m \cdot \ln m}\ \right)$

Theorem
There exists a randomized voting rule with distortion $O(\sqrt{m} \log^* m)$

- But, not that much better …

Theorem
The distortion of any randomized voting rule is $\Omega(\sqrt{m})$

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- **Deterministic rules:** distortion is $\Omega(m^2)$

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- These values induce the preference rankings
  - Many different valuation profiles can induce the same ordinal profile
- **Distortion:** worst case ratio over all valuation profiles between the social welfare of the optimal outcome over the social welfare of the outcome chosen by the voting rule
- **Deterministic rules:** distortion is $\Omega(m^2)$
- **Randomized rules:** distortion is between $\Omega(\sqrt{m})$ and $O(\sqrt{m}\log^* m)$

Some further readings

- **The distortion of cardinal preferences in voting**
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